Math 250 4.1 Maxima and Minima

Extrema, or extreme values, refer to high and low y-values of a function.
Singular: a minimum, a maximum, an extremum
Plural: several minima, several maxima, several extrema

Objectives:

1) **Recognize the difference between a relative extreme value and an absolute extreme value.**
2) **Find critical values.**
3) **Use critical values and endpoints to find extrema on a closed interval.**

Local or Relative Extrema: The point is highest or lowest compared to nearby points.

A function *f* has a relative (or local) maximum at *c* if $f(c) \geq f(x)$ for all *x* in some open interval containing *c*.
A function *f* has a relative (or local) minimum at *c* if $f(c) \leq f(x)$ for all *x* in some open interval containing *c*.
A function *f* has a relative (or local) extremum at *c* if the function has either a relative maximum or minimum at *c*.

Global or Absolute Extrema: The point is highest or lowest on the function over its entire domain.

A function *f* has an absolute (or global) maximum at *c* if $f(c) \geq f(x)$ for all *x* in the domain of *f*.
A function *f* has an absolute (or global) minimum at *c* if $f(c) \leq f(x)$ for all *x* in the domain of *f*.
A function *f* has an absolute (or global) extremum at *c* if the function has either an absolute maximum or minimum at *c*.

Critical values (or critical numbers) of a function are $x = c$ so that

- *c* is in the domain of *f*, meaning that $f(c)$ is defined and finite
- *c* is not an endpoint of the domain
- $f'(c) = 0$ or $f'(c)$ is undefined, meaning that the tangent to the graph at $x = c$ is vertical or horizontal.

CAUTION: A value *c* where $f(c)$ is undefined is not a critical value! However, when we make sign charts and analyze graphs (in other sections), we must include values of *c* where $f(c)$ is undefined, even though these cannot be extrema.

To find absolute extrema of a continuous function on a closed interval:

1. Confirm that the function is continuous on the interval of interest – no holes, jumps, or vertical asymptotes.
2. Find the critical values of the function.
3. Select the critical values which are within the given interval. (Discard others.)
4. Evaluate function values for the critical values.
5. Evaluate the function at the endpoints of the interval.
6. Identify the largest function value as the maximum.
7. Identify the smallest function value as the minimum.

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Vocabulary

minimum value of a function = lowest y coordinate

minima = more than one minimum

maximum value of a function = highest y coordinate

maxima = more than one maximum

extremum = either a maximum or a minimum

extrema = more than one extremum

Note: to be a maximum or minimum value, the y-coordinate must be finite. (The function is defined.)

If the highest y-value is $+\infty$, we say it has no maximum.

If the lowest y-value is $-\infty$, we say it has no minimum.

Note #2: The x-coordinate corresponding to the extremum is the location of the extreme value.

① Ex: "The function has a max of 5 at $x = 3$."
means $f(3) = 5$, or $(3,5)$ is the highest point on the graph.

CAUTION: "Find all extrema" means ...

- find x-values (locations)
- find y-values (extrema)
- test to confirm they're extreme values

★ identify whether it's a maximum or a minimum.

Usually the instructions will ask for "relative/local" or "absolute/global", but if not....

- identify which type of extremum.

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Global or Absolute Extremum:

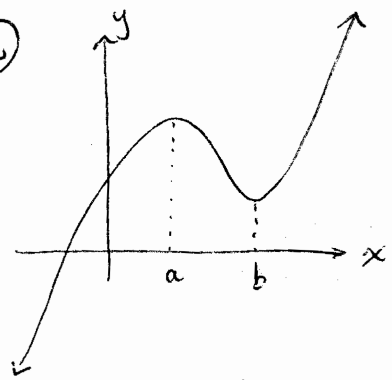
The largest/smallest y-value of the function
- over its entire domain
- over an interval given to be its entire domain.

Local or Relative Extremum:

The largest/smallest y-value of a function
- compared to its nearby neighbors

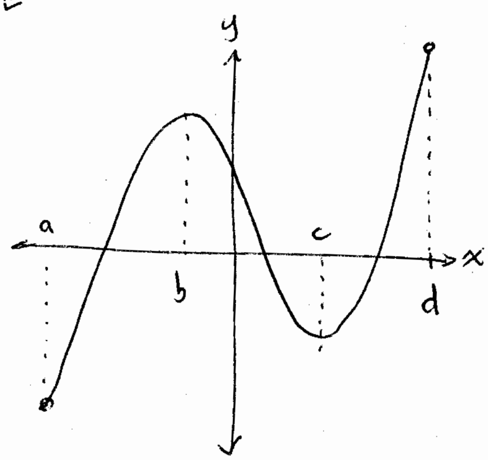
For each graph, identify
- locations of absolute maxima
- locations of absolute minima
- locations of relative maxima
- locations of relative minima
- values of f' at each interior value

(2)



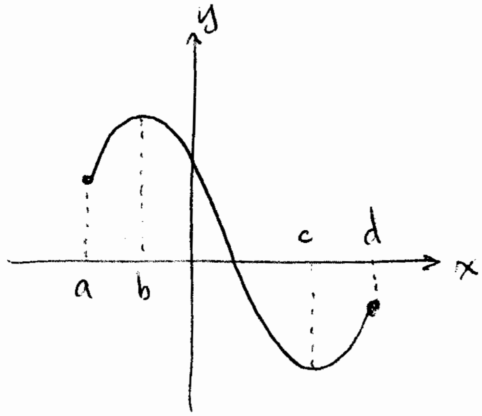
no absolute max
no absolute min
relative max $f(a)$ at $x=a$, $f'(a)=0$
relative min $f(b)$ at $x=b$, $f'(b)=0$
(horizontal tangents)

(3)



abs max $f(d)$ at $x=d$
abs min $f(a)$ at $x=a$
rel max $f(b)$ at $x=b$ $\quad f'(b)=0$
rel min $f(c)$ at $x=c$ $\quad f'(c)=0$
(horizontal tangents)

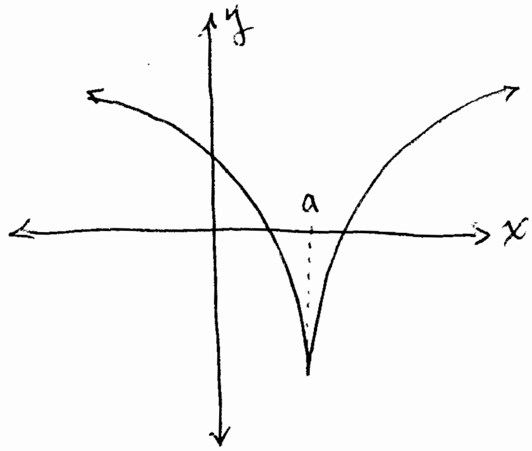
(4)



abs and rel max $f(b)$ at $x=b$ $\quad f'(b)=0$
abs and rel min $f(c)$ at $x=c$ $\quad f'(c)=0$

(horizontal tangents)

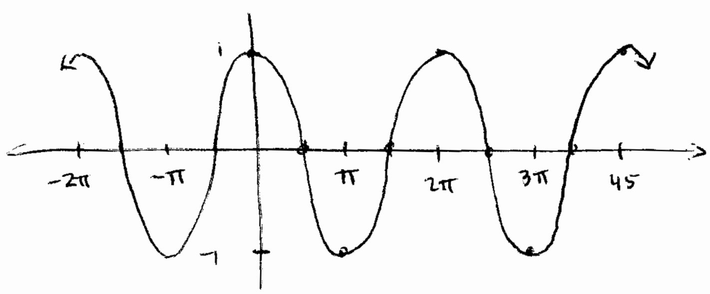
(5)



no abs or rel max
abs and rel min $f(a)$ at $x=a$
$f'(a)$ undefined

(vertical tangent)

(6) $f(x) = \cos(x)$



abs and rel max 1
at $x = 2\pi k \quad k \in \mathbb{Z}$
$f'(2\pi k) = 0$

abs and rel min -1
at $x = (2k-1)\pi \quad k \in \mathbb{Z}$
$f'((2k-1)\pi) = 0$

(horizontal tangents)

Note: A function may have many locations x associated with the same max (or min) value

Definitions: A critical value $x=a$ is such that
- $f(a)$ is defined $\rightarrow$ finite, on the graph (real)
- $f'(a) = 0$ OR $f'(a)$ undefined $\rightarrow$ horizontal or vertical tangent
- $x=a$ is not an endpoint $\rightarrow$ "interior point".

A critical point $(a, f(a))$ is an ordered pair where $x=a$ is a critical value.

("critical value" is sometimes called "critical number".)

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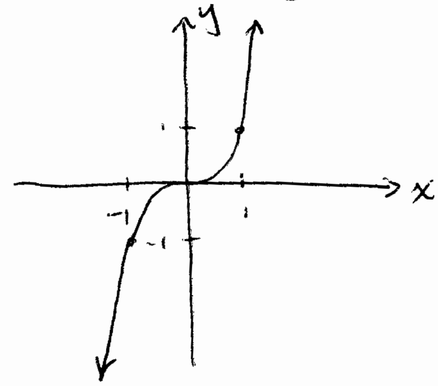
(7) Find the critical values of $f(x) = x^5$. Sketch graph.

$f'(x) = 5x^4$

$f'(x) = 0 \Rightarrow 5x^4 = 0$

$\boxed{x = 0}$

$f'(x)$ never undefined.



Note: $x=0$ is a critical value but $f(0)$ is not an extremum!

Consider these statements logically –

(I.) If $x=c$ is a critical value of f then $f(c)$ is a relative extremum. → No, see (7) $f(x) = x^5$

(II) If $x=c$ is a critical value of f then $f(c)$ is an absolute extremum. → No, see (1) or (2)

(III) If $f(c)$ is an absolute extremum Then $x=c$ is a critical value → No, see (4)

(IV) If $f(c)$ is a relative extremum Then $x=c$ is a critical value. → YES!

Critical values (C.V.s) can occur at values of x where there is no extremum.

But relative extrema must occur at C.V.s.

<u>So</u>: We'll find C.V.s, but we will always test them

- See if they're extreme value location
- identify which type of extremum: maximum or minimum.

In 4.1: We want <u>absolute</u> extrema only and only when an interval is given.

Process:

<u>Step 1</u>: find $f'(x)$

<u>Step 2</u>: find CVs $f'(x)=0$ & $f'(x)$ undef

<u>Step 3</u>: Make a table of y values for CVs <u>and</u> endpoints.

Step 4: State conclusions.

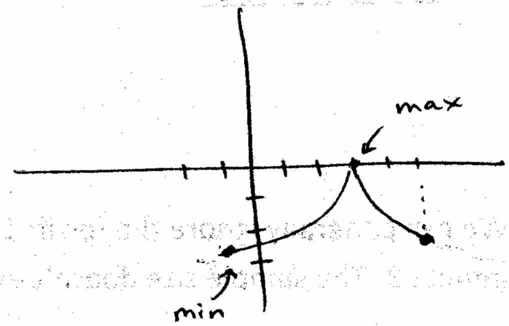
Math 250 Find absolute extrema on the given interval.

(8) $g(x) = -(x-3)^{2/3}$ on $[-1, 5]$

critical values

$g'(x) = -2/3(x-3)^{-1/3}$

$g'(x) = \dfrac{-2}{3\sqrt[3]{x-3}}$



$g'(x) \neq 0$ at any x

$g'(x)$ undefined when $\sqrt[3]{x-3} = 0$

$x - 3 = 0$

$x = 3$ critical value

Use decimals to decide

x	y
-1	-2.52
3	0
5	-1.587

con't →

$g(-1) = -(-1-3)^{2/3}$

$= -(-4)^{2/3}$

$= -\sqrt[3]{(-4)^2}$

$= -\sqrt[3]{16}$ $16 = 2^3 \cdot 2$

$= -2\sqrt[3]{2}$

$\boxed{\begin{array}{l} \text{max } 0 \text{ at } x = 3 \\ \text{min } -2\sqrt[3]{2} \text{ at } x = -1 \end{array}}$

Give exact answers.

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9 $f(x) = \dfrac{2x}{x^2+1}$ on $[-2,2]$

critical values

$$f'(x) = \frac{(x^2+1)\cdot 2 - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$= \frac{-2x^2+2}{(x^2+1)^2}$$

$$= \frac{-2(x^2-1)}{(x^2+1)^2}$$

$f'(x) = 0$ if numerator $= 0$ $\quad -2(x+1)(x-1) = 0$

$x = 1, -1$

$f'(x)$ undef if denominator $= 0$ $\quad (x^2+1)^2 = 0$

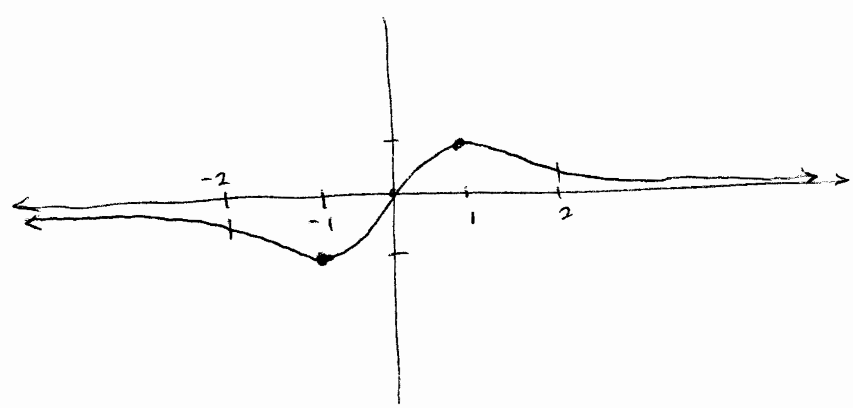
$x^2+1 = 0$

$x^2 = -1$

imaginary

x	y
-2	-.8
1	1
2	.8
-1	-1

$\boxed{\begin{array}{l}\text{min } -1 \text{ at } x=-1 \\ \text{max } 1 \text{ at } x=1\end{array}}$



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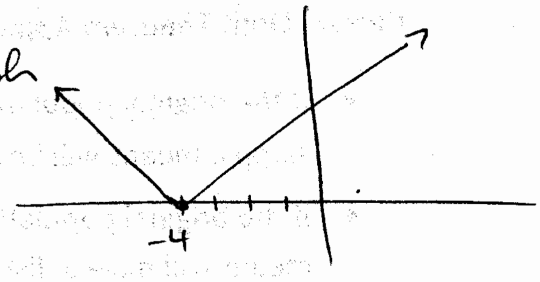
(10) $f(x) = |x+4|$ on $[-7, 1]$

critical values: remember graph



$f'(x)$ DNE at $x = -4$

because $m_{\text{TAN Left}} \neq m_{\text{TAN Right}}$.

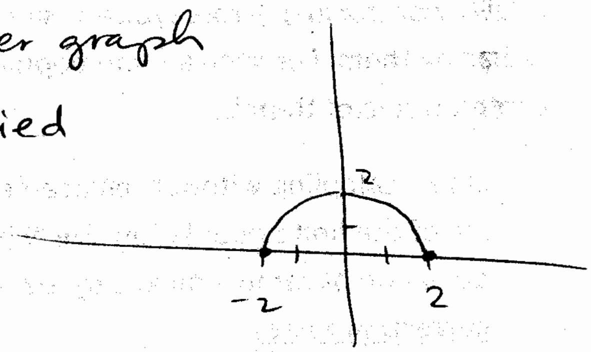
x	y
-7	3
-4	0
1	5

$\boxed{\text{max } 5 \text{ at } x=1 \quad \text{min } 0 \text{ at } x=-4}$

(11) $f(x) = \sqrt{4-x^2}$

critical values: remember graph

This function has an implied domain $[-2, 2]$



There should also be a horizontal tangent

$$f'(x) = \tfrac{1}{2}(4-x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$f'(x) = 0$ when $\underset{=0}{\text{numerator}}$ $\quad -x = 0$

$x = 0$

$f'(x)$ DNE when denom $= 0$ $\quad \sqrt{4-x^2} = 0$

$4 - x^2 = 0$

$x = \pm 2$

x	y
-2	0
0	2
2	0

$\boxed{\text{min } 0 \text{ at } x=2 \text{ and } x=-2 \quad \text{max } 2 \text{ at } x=0}$

12 1. **(a)** Find the critical points, if any, of the following function on the given interval.
(b) Determine the absolute extreme values of f on the given interval.
(c) Use a graphing utility to confirm your conclusions.

Use this example

$f(x) = 4\cos^2 x$ on $[0,\pi]$

(a) Find the critical points, if any, of $f(x) = 4\cos^2 x$ on $[0,\pi]$. Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

- **A.** The critical point(s) occur(s) at x = $\frac{\pi}{2}$.
  (Type an exact answer, using π as needed. Use a comma to separate answers as needed.)
- **B.** There are no critical points for f(x) on $[0,\pi]$.

(b) Determine the absolute extreme values of $f(x) = 4\cos^2 x$ on $[0,\pi]$. Select the correct choice below and, if necessary, fill in the answer boxes to complete your choice.

- **A.** The absolute maximum is ______ at x = ______ and the absolute minimum is ______ at x = ______.
  (Type an exact answer, using π as needed. Use a comma to separate answers as needed.)
- **B.** There are no absolute extreme values for f(x) on $[0,\pi]$.

YOU ANSWERED: A.: [0]

$f'(x) = 8\cos x \cdot (-\sin x)$

$f'(x) = -8\cos x \sin x = 0$

$\cos x = 0 \qquad \sin x = 0$

$x = \frac{\pi}{2} \qquad x = 0, \pi \qquad$ critical points $0, \frac{\pi}{2}, \pi$

except defn says interior pts, hence $\frac{\pi}{2}$ only.

x	f(x)
0	4
$\pi/2$	0
π	4

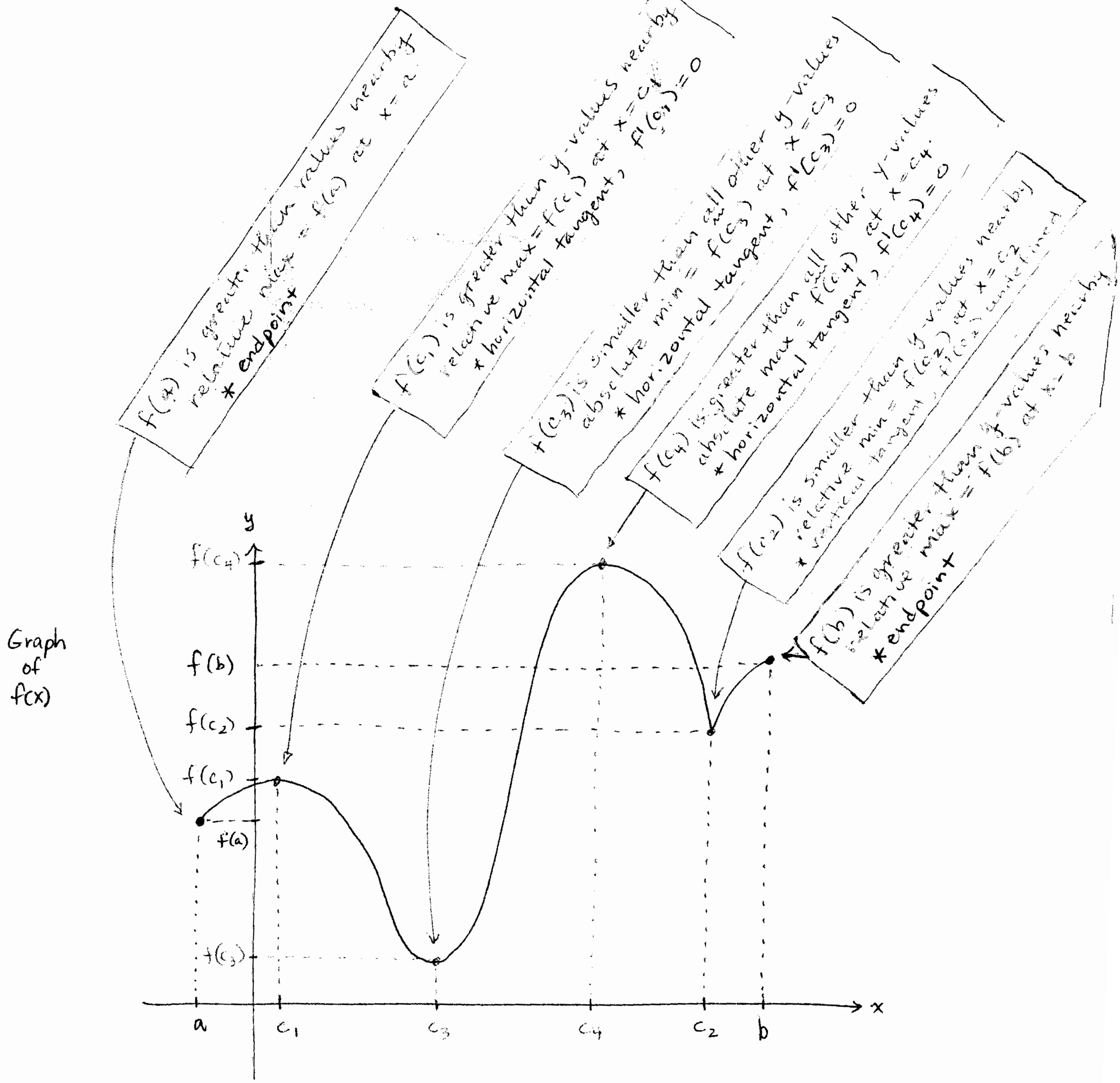
$4\cos^2(0) = 4(1)^2 = 4$

$4\cos^2(\frac{\pi}{2}) = 4(0)^2 = 0$

$4\cos^2(\pi) = 4(-1)^2 = 4$

absolute max is 4 at $x = 0, \pi$
absolute min is 0 at $x = \frac{\pi}{2}$

Absolute (Global) Extrema and Relative (Local) Extrema



Extra Examples

① step-by-step

Find all extrema.
Specify if relative or absolute.

$f(x) = x^2$ on $[-1, 2]$.

Step 1: Find $f'(x)$ by differentiating.

$f'(x) = 2x$

Step 2: Find values of x where $f'(x)$ undefined.

none

Step 3: Find values of x where $f'(x) = 0$

$2x = 0$

$x = 0$

Step 4: Open or closed interval?

closed → must consider endpts.
(open → ignore endpoints)

Step 5: Make a table of endpts & CVs.

	x	$y = f(x)$	
$f' \rightarrow$	0	$0^2 = 0$	← smallest
end pts	-1	$(-1)^2 = 1$	
end pts	2	$2^2 = 4$	← largest

Step 6: Conclusions.

$\boxed{\begin{array}{l}\text{abs min } 0 \text{ at } x = 0 \\ \text{abs max } 4 \text{ at } x = 2\end{array}}$

Math 250 Extra Examples

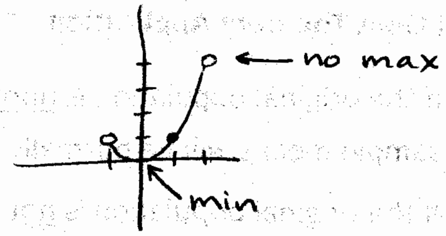
(2) $f(x) = x^2$ on $(-1, 2)$

critical value $x = 0$

no endpts

	x	y
c.v.	0	0

$\boxed{\text{min } 0 \text{ at } x = 0}$



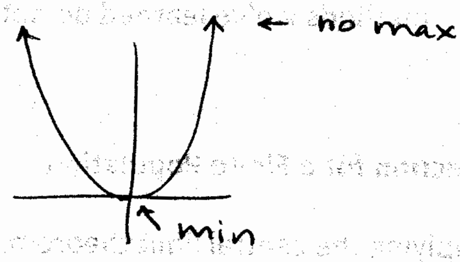
(3) $f(x) = x^2$ on $(-\infty, \infty)$

critical value $x = 0$

no endpts

	x	y
c.v.	0	0

$\boxed{\text{min } 0 \text{ at } x = 0}$

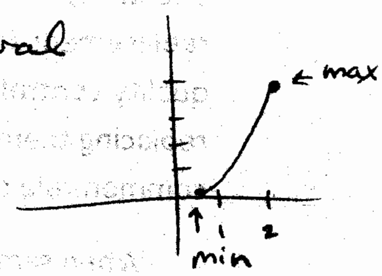


(4) $f(x) = x^2$ on $[\frac{1}{2}, 2]$

critical value $x = 0$ is not within interval

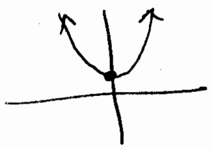
	x	y
endpt	$1/2$	$1/4$
endpt	2	4

$\boxed{\begin{array}{l} \text{min } 1/4 \text{ at } x = 1/2 \\ \text{max } 4 \text{ at } x = 2 \end{array}}$



Math 250 Extra Examples:

(5) $f(x) = x^2 + 1$ graph on entire domain.

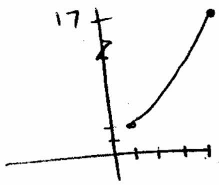


absolute min $(0,1)$
= relative min $(0,1)$

no max

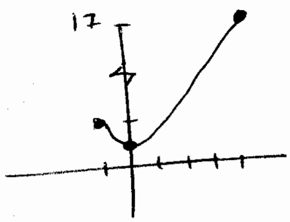
$f'(x) = 2x$
at $x = 0$
$f'(0) = 0$
horizontal tangent line.

(6) $f(x) = x^2 + 1$ on $[1,4]$



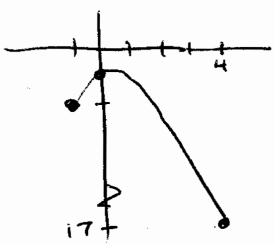
min $(1,2)$
max $(4,17)$ — endpoints.

(7) $f(x) = x^2 + 1$ on $[-1,4]$



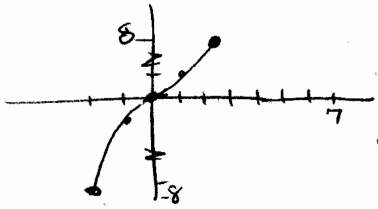
min $(0,1)$ — horizontal tangent
max $(4,17)$ — endpoint

(8) $f(x) = -x^2 - 1$ on $[-1,4]$



max $(0,-1)$ — horizontal tangent
min $(4,-17)$ — endpoint.

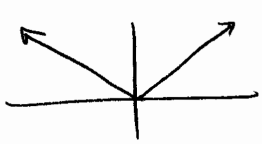
(9) $f(x) = x^3$ on $[-2,2]$



max $(2,8)$
min $(-2,-8)$

$f'(x) = 3x^2$
$f'(0) = 0$
but $(0,0)$ is neither max nor min!

(10) $f(x) = |x|$



min $(0,0)$
no max (global - no interval)

$f'(x)$ undef at $x = 0$
$\Rightarrow$ important point!!

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

slope of tangent line used to determine deriv.

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Find the extrema on the given (closed) interval.

(11)

$f(x) = x^2+1$ on $[-1,2]$

$f'(x) = 2x$

$2x = 0$

$x = 0$ critical value

[$f'(x)$ defined everywhere]

x	$f(x)$
0	1
-1	2
2	5

abs. minimum	$(0,1)$
rel. maximum	$(2,5)$

(12)

$f(x) = x^3 - 3x^2$ on $[-1,3]$

$f'(x) = 3x^2 - 6x$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0, 2$ critical values

[$f'(x)$ defined everywhere].

x	$f(x)$
0	0
2	-4
-1	-4
3	0

maxima	$(0,0)$ and $(3,0)$
minima	$(2,-4)$ and $(-1,-4)$

$f(x) = x^3 - 3x^2$ on $(-1,3)$

$x = 0, 2$ C.Vs.

x	$f(x)$
0	0
2	-4

max 0 at $x=0$

min -4 at $x=2$

(13)

$f(x) = -(x-3)^{2/3}$ on $[0,4]$

$f'(x) = -\frac{2}{3}(x-3)^{-1/3}$

$= \dfrac{-2}{3\sqrt[3]{x-3}}$

$f'(x) \neq 0$ anywhere

cont

$f'(x)$ undef when $x-3=0$

$x=3$. critical value

x	$f(x)$
3	0
0	≈ -2.08
4	-1

$\longrightarrow f(0) = -(0-3)^{2/3}$

$= -(9)^{1/3}$

$= -\sqrt[3]{9}$

maximum	$(3,0)$
minimum	$(0,-\sqrt[3]{9})$

[Takes 25 min] [skip to next]

(14) $f(x) = 2\sin x + \cos 2x$ on $[0, 2\pi]$

$f'(x) = 2\cos x - \sin 2x \cdot 2$

$= 2\cos x - 2\sin 2x.$

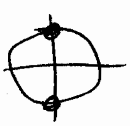
$2\cos x - 2\sin 2x = 0$

use double-angle identity
$\sin 2x = 2\sin x \cos x.$

$2\cos x - 2(2\sin x \cos x) = 0$

$2\cos x - 4\sin x \cos x = 0$

$2\cos x(1-2\sin x) = 0$ factor



$2\cos x = 0$

$\cos x = 0$

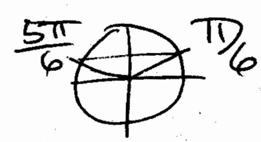
$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$1-2\sin x = 0$

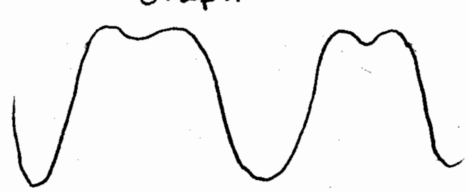
$-2\sin x = -1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$



Graph:



x	$f(x)$
$\pi/2$	1
$3\pi/2$	-3
$\pi/6$	1.5
$5\pi/6$	1.5
0	1
2π	1

minimum	$(\frac{3\pi}{2}, -3)$
maxima	$(\frac{\pi}{6}, \frac{3}{2})$
	$(\frac{5\pi}{6}, \frac{3}{2})$